

Charged-spinning-gravitating Q-balls

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Abstract

We consider the lagrangian of a self-interacting complex scalar field admitting generically Q-balls solutions. This model is extended by minimal coupling to electromagnetism and to gravity. A stationnary, axially-symmetric ansatz for the different fields is used in order to reduce the classical equations. The system of non-linear partial differential equations obtained becomes a boundary value problem by supplementing a suitable set of boundary conditions. We obtain numerical evidences that the angular excitations of uncharged Q-balls, which exist in flat space-time, get continuously deformed by the Maxwell and the Einstein terms. The electromagnetic and gravitating properties of several solutions, including the spinning Q-balls, are emphasized.

1 Introduction

In the quest for new theoretical objects which could constitute a fraction of the exotic component (see e.g. [1]) of the Universe, Q-balls [2, 3, 4] and their gravitating counterparts, boson stars [5, 6, 7], emerge as serious candidates. Many efforts are carried out for an experimental detection of boson stars [8], but no evidence of their presence has been obtained so far. From the phenomenological point of view, the models containing the basic ingredients for Q-balls (i.e. scalar fields) are looked for into the supersymmetric extensions of the standard model of particle physics [9, 10, 11, 12]. The super-partners of the quarks and leptons naturally lead to scalar fields. However, there is a lot of incertitude in chosing an appropriate model.

Far away from cosmological considerations and from these elaborated lagrangians, several basic features of scalar fields in interaction with gravity and/or electromagnetism can be studied in more elementary models. Here the basic field is a complex scalar field self-interacting through an appropriate potential. The spectrum of Q-balls solutions turns out to be extremely rich, the solutions present an harmonic time dependance characterized by a 'spectral'-parameter ω_s . Besides the spherically symmetric (fundamental) solutions, radially excitated solution exist as well[13]. The solutions can be done spinning both with even and odd branches [13] under the

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parity operator. Angular excitations of Q-balls in relations with the spherical harmonics are discussed in [14]. Spinning solutions in supersymmetric extensions of the standard model are constructed in [15, 16].

Once coupling the lagrangian admitting Q-balls to gravity, the solutions get continuously deformed by gravity but lead to an even more involved pattern [17, 18]. Several branches occur, existing on finite intervals of the parameter ω_s , and terminating into cusps at a series of critical values $\omega_s = \omega_{s,k}$, $k = 1, 2, \dots$.

Q-balls are intimately related to a global U(1) symmetry. The gauging of this symmetry naturally introduces the electromagnetic field into the lagrangian. It is therefore challenging to understand the pattern of solutions to both gravitation and electromagnetism. The electromagnetic properties of spinning Q-balls are considered in [19], Charged, spherically symmetric boson stars are studied in [20] for a very specific potential of the scalar field. More generic potentials are proposed e.g. in [22]. To our knowledge, the effects of the electromagnetic fields on the angular excitations of boson star, including the important case of spinning solutions have not yet been addressed in the litterature. The purpose of this paper is to investigate such solutions.

In the second section of this paper, we consider the Einstein-Maxwell Lagrangian coupled to the Lagrangian of a scalar field admitting the basic Q-balls as generic solutions. We present the axially symmetric ansatz and discuss the physical quantities characterizing the solutions.

We sketch the classical equations and the set of relevant boundary conditions in Sect. III. These equations were solved by using numerical methods, several results and properties of the numerical solutions that we obtained are presented. The solutions can be characterized by mass, particle number, angular momentum, electric charge and magnetic moment. In passing, we note, that the configurations obtained can be seen as regularized version of the Kerr-Newman black holes where the horizon and the essential singularity occuring at the origin are regularized by the scalar field.

2 The model

The starting Lagrangian describes a complex scalar field in $3 + 1$ dimensions admitting Q-balls solutions. We then couple the scalar field to an electromagnetic field in the standard way, obtaining an abelian gauge theory. The resulting Klein-Gordon-Maxwell Lagrangian is then coupled minimally to gravity. The full action S reads:

$$S = \int \sqrt{-g} d^4x \left(\frac{R}{16\pi G} + \mathcal{L}_m \right) \quad (1)$$

where R is the Ricci scalar, G represents Newton's constant and \mathcal{L}_m denotes the matter Lagrangian:

$$\mathcal{L}_m = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\Phi D^\mu\Phi^* - V(\Phi) \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2)$$

Here Φ denotes the complex scalar fields and $D_\mu = (\partial_\mu - ieA_\mu)$ is the covariant derivative with a coupling constant e . The signature of the metric is $(-+++)$. The conventional form of the self-interacting potential is used:

$$V(\Phi) = \kappa|\Phi|^6 - \beta|\Phi|^4 + \lambda|\Phi|^2 \quad (3)$$

where κ, β, λ , are positive constants. It was argued in [13] that a Φ^6 -potential is necessary for classical Q -ball solutions to exist. The parameter λ determines the mass of the scalar field : $(m_B)^2 = \lambda$.

With the purpose of comparing our results with those existing in the litterature [17, 18, 14] in the limit $e = 0$, the values

$$\kappa = 1 \quad , \quad \beta = 2 \quad , \quad \lambda = 1.1 \quad . \quad (4)$$

are used in the explicit calculations.

The energy-momentum tensor reads:

$$T_{\mu\nu} = (D_\mu\Phi)^*(D_\nu\Phi) + (D_\nu\Phi)^*(D_\mu\Phi) + F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} + g_{\mu\nu}\mathcal{L}_m \quad (5)$$

The conserved Noether current j^μ , $\mu = 0, 1, 2, 3$, associated to the U(1)- symmetry is just

$$j^\mu = -i(\Phi^*D^\mu\Phi - \Phi D^\mu\Phi^*) \quad (6)$$

leading to a conserved charge Q of the system

$$Q = -i \int j^t \sqrt{-g} dr d\theta d\varphi \quad (7)$$

The variation of the action 1 with respect to various degrees of freedom, namely the metric, the scalar and vector fields leads respectively to the Einstein equations

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \quad (8)$$

(with $T_{\mu\nu}$ given by (5)), the Klein-Gordon equations

$$\left(\square + \frac{\partial V}{\partial|\Phi|^2}\right)\Phi = 0 \quad , \quad (9)$$

and the Maxwell equations

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = \sqrt{-g} \frac{e}{2}(\Phi^*D^\nu\Phi + D^\nu\Phi\Phi^*) \quad . \quad (10)$$

3 The Q-ball equations

3.1 Ansatz

We will look for stationnary, axially-symmetric solutions along the direction z . For the metric, we use Lewis-Papapetrou coordinate frame [17]:

$$ds^2 = -f dt^2 + \frac{L}{f} \left(g(dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta (d\varphi + \frac{w}{r} dt)^2 \right) \quad (11)$$

where the metric functions f , L , g and w are functions of r and θ (N.B. we use the notation L instead of the conventional one l to avoid confusion in the next sections). This ansatz is completed with the following form for the scalar and electromagnetic fields :

$$\Phi(t, r, \theta, \varphi) = e^{i\omega_s t + ik\varphi} \phi(r, \theta) \quad , \quad \mathcal{A} = V(r, \theta) dt + A(r, \theta) \sin \theta d\varphi \quad (12)$$

depending on three new functions ϕ, V, A while ω_s and k are constants. The periodicity of the scalar field $\Phi(\varphi) = \Phi(\varphi + 2\pi)$ requires $k \in \mathbb{Z}$.

Inserting the ansatz in the equations (8),(9),(10) leads to a system of seven non-linear partial differential equations given in appendix. These equations depend on the spectral parameter ω_s , on the integer k and on the coupling constants e, G . It is convenient to use dimensionless variables and functions. In this purpose, we set

$$\hat{r} = M_p r \quad , \quad \hat{\omega}_s = \frac{\omega_s}{M_p} \quad , \quad \hat{\phi} = \frac{\phi}{M_p} \quad , \quad \hat{V} = \frac{V}{M_p} \quad , \quad \hat{A} = \frac{A}{M_p} \quad , \quad (13)$$

where M_p is a mass scale associated with the phenomenology of the underlying model (see e.g. [21],[16] for details). We then use $\alpha \equiv 8\pi G M_p^2$ and, for convenience, we drop the hat on the dimensionless quantities in the rest of the paper. The potential is fixed by (4). For $e = 0$, the Maxwell equations decouple, the remaining equations are those of the (uncharged) boson star. For $k = 0$, the equations for the fields w and A are solved trivially by $w = 0, A = 0$.

Several limits of these equations have been studied over the recent years.

- The case $\alpha = e = 0$ was the object of [13].
- The case $e = 0$ was studied in [17, 18] and in [14].
- Charged-rotating solutions (i.e. $\alpha = 0, e \neq 0, k = 1$) are constructed in [19]
- The case $k = 0$, corresponding to spherically symmetric charged boson star, was adresssed in [20].

In [14], it was argued that different angular excitations of (uncharged) Q-balls and boson-stars exist in correspondance with the symmetries of the spherical harmonic functions $Y_\ell^k(\theta, \varphi)$, with $-\ell \leq k \leq \ell$. The fundamental solutions correspond to $\ell = k = 0$. It is therefore natural to characterize the different families of solutions (and their corresponding charged generalisations) by using the integers ℓ, k .

3.2 Boundary conditions

We are looking for regular, stationary solutions and presenting a symmetry (or an antisymmetry) under the reflexion $z \rightarrow -z$. The domain of integration can then be limited to $r \in [0, \infty], \theta \in [0, \pi/2]$. It turns out that the boundary conditions for the function ϕ depend on the discrete symmetries of the fields, i.e. on the numbers ℓ, k . Here we give the ones corresponding to $\ell = 0, 1$ only (higher values can be found in [24]). The appropriate boundary conditions read:

$$r = 0 : \partial_r f = 0, \quad \partial_r L = 0, \quad g = 1, \quad w = 0, \quad \phi = 0, \quad \partial_r V = 0, \quad A = 0. \quad (14)$$

for solutions with $k \neq 0$, while for $k = 0$ solutions, we have $\partial_r \phi|_{r=0} = 0$. The boundary conditions at infinity result from the requirement of asymptotic flatness and finite energy solutions:

$$r = \infty : f = 1, \quad L = 1, \quad g = 1, \quad w = 0, \quad \phi = 0, \quad V = 0, \quad A = 0. \quad (15)$$

For $\theta = 0$ the regularity of the solutions on the z -axis requires:

$$\theta = 0 : \partial_\theta f = 0, \quad \partial_\theta L = 0, \quad g|_{\theta=0} = 1, \quad \partial_\theta w = 0, \quad \phi = 0, \quad \partial_\theta V = 0, \quad A = 0. \quad (16)$$

for $k \neq 0$ solutions, while for $k = 0$ solutions, we have $\partial_\theta \phi|_{r=0} = 0$.

The conditions at $\theta = \pi/2$ are either given by

$$\theta = \frac{\pi}{2} : \partial_\theta f = 0, \quad \partial_\theta L = 0, \quad \partial_\theta g = 0, \quad \partial_\theta w = 0, \quad \partial_\theta \phi = 0, \quad \partial_\theta V = 0, \quad \partial_\theta A = 0, \quad (17)$$

for even parity solutions, while for odd parity solutions the conditions for the scalar field functions read: $\phi|_{\theta=\pi/2} = 0$.

3.3 Physical quantities

The boson stars can be characterized by several physical parameters. The mass M and total angular momentum J of the solution can be computed from the appropriate Komar integral; it was shown in [17] that these conserved charges can finally be read off from the asymptotic behaviour of the metric functions :

$$M = \frac{1}{2G} \lim_{r \rightarrow \infty} r^2 \partial_r f, \quad J = \frac{1}{2G} \lim_{r \rightarrow \infty} r^2 w. \quad (18)$$

The total angular momentum J and the Noether charges Q of the two boson stars are related according to $J = kQ$ as pointed out first in [23]. Boson stars with $k = 0$ have thus vanishing angular momentum. The goal of this paper is to emphasize the electromagnetic properties of the solutions. The electric charge Q_e and magnetic moment μ are given by the asymptotic decay of the electromagnetic potentials [19], i.e.

$$V = \frac{Q_e}{4\pi r} \quad , \quad A = -\frac{\mu}{4\pi r} \quad , \quad \text{for } r \rightarrow \infty \quad (19)$$

and the gyromagnetic factor g can then be defined : $\mu = gQJ/(2M)$.

4 Numerical results

We have solved numerically the system of partial differential equations (8) and (9) subject to the appropriate boundary conditions given in Section 3.2. This has been done using the PDE solver FIDISOL [25]. We have mapped the infinite interval of the r coordinate $[0 : \infty]$ to the finite compact interval $[0 : 1]$ using the new coordinate $z := r/(r + 1)$. We have typically used grid sizes of 150 points in r -direction and 70 points in θ direction. The solutions presented below have relative errors of order 10^{-3} or smaller.

4.1 Non-rotating solutions

This corresponds to $k = 0$. In this case, the system of differential equations reduces to a system of coupled ordinary differential equations. Four of the equations are solved trivially by $g \equiv 1$, $L = 1$, $w \equiv 0$ and $A = 0$, implying a null magnetic field. The functions f, ϕ, V depend on the radial variable r only. In the uncharged case (i.e. with $e = 0$), the gravitating Q-balls corresponding to the potential (3),(4) exist on a finite interval of the parameter ω_s , i.e. for $\omega_s \in [\omega_1, \omega_2]$. Several supplementary branches of solutions further develop in the region of $\omega_s \sim \omega_a$, forming a set of backbending branches stopping in cusp at several critical values of ω_s . We expect this pattern of solutions to be preserved for $e > 0$. The evolution of the mass and of the conserved charge Q as functions of the electric charge Q_e is reported on Fig. 1 for $\omega_s = 0.85$ and $\alpha = 0.1, \alpha = 0.5$. For these values, there is only one uncharged solution [17]. Because charged spherically symmetric solutions were first addressed in [20] (although with a potential different from (3)), we put in this paper the main emphasis on several kind of angular excitations, especially to the important case of spinning solutions corresponding with $k = 1$.

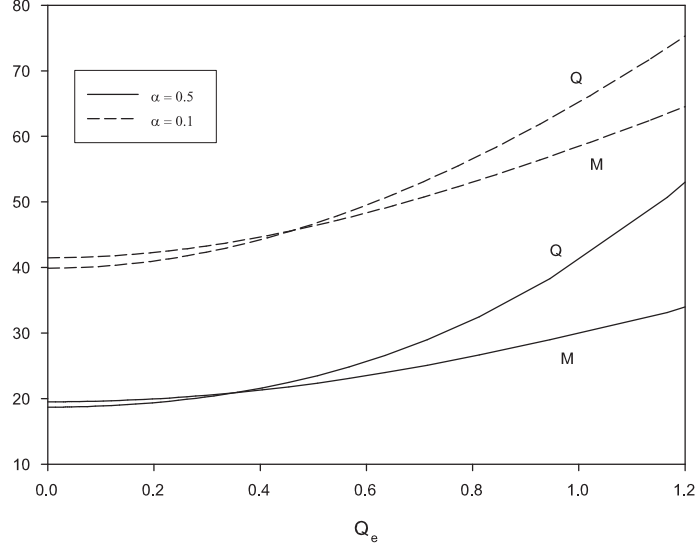


Figure 1: Values of M and Q as functions of the electric charge Q_e for $\alpha = 0.1$ and $\alpha = 0.5$ for the spherically symmetric boson star with $\omega_s = 0.85$

4.2 Spinning solutions

Setting $k = 1$, the solutions are charged-spinning-gravitating Q-balls. It was shown in [17] that several branches of uncharged spinning solutions exist once $\alpha > 0$. Two branches of solutions exist on a large interval of the parameter ω_s , but additional branches further develop with a smaller extension of the parameter ω_s . The equations depend on three continuous parameters: α, e, ω_s . Exploring these three independent directions represent a huge task which is not the scope of this paper. We therefore limited our investigation to $\omega = 0.75$ and $\alpha \in [0.1, 0.5]$ for which there are two branches of uncharged spinning boson stars [17]. Once increasing gradually the electric coupling constant e the solutions naturally get charged and develop both an electric and a magnetic fields. The two uncharged solutions form two branches, each labelled by the electric charge Q_e . Examining first the case of weak gravity, it seems that the two branches can be continued for arbitrarily large values of the electric charge. The mass and angular momentum of the solutions is represented on Fig. 2 as functions of the electric charge Q_e . The magnetic momentum μ and the corresponding constant e are supplemented in the window of the figure. Considering then the spinning solutions for large values of α leads to a different scenario. It turns out, indeed, that the two branches exist up to a maximal value of the electric charge, say $Q_e = Q_e^{max}$. Once this limit is attained the two branches coincide and it is very likely that no solution exist for $Q_e > Q_e^{max}$. In the case $\omega_s = 0.75, \alpha = 0.5$, we find $Q_e^{max} \approx 1.9$. This is

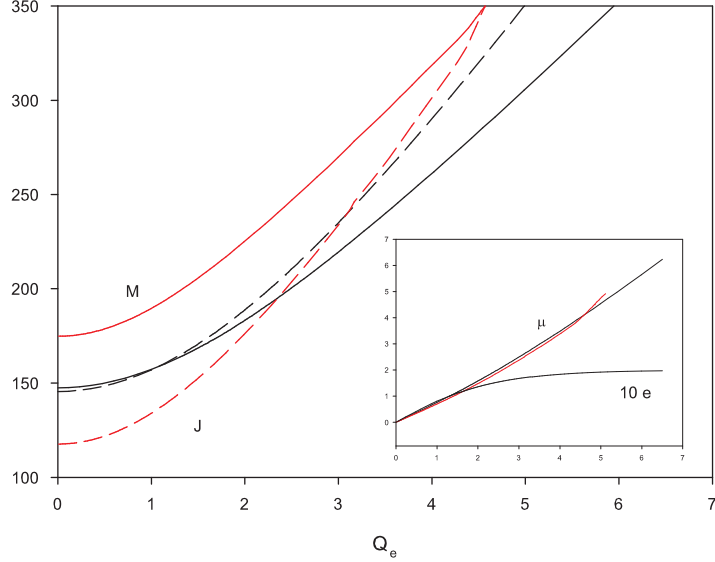


Figure 2: Mass M and angular momentum J of the $\ell = k = 1$ Q-ball as functions of the electric charge for $\alpha = 0.1$ and $\omega_s = 0.75$

illustrated by Fig. 3; we see that the electric charge links the branches of configuration emerging from the two (uncharged) solutions of [17]. As an attempt to interpret this phenomenon, we remember that, for a given value of ω_s , the mass of the soliton decreases for increasing α . This suggests that the mass of the soliton becomes too weak for the attractive gravitational force to dominate the electrostatic repulsion inside the soliton.

The contour plots of a typical charged-spinning solution in the ρ, z plane are presented in Fig.4 for $\alpha = 0.5$. It reveals that the functions ϕ, w, A are concentrated in a torus lying in the equator plane, while f, V deviate only a little from spherically symmetric configurations.

4.3 Angular excited boson stars

Radial excitations of Q-balls have been emphasized for a long time [13], the argument demonstrating their existence is based on the analogy between the classical movement of a point particle into a potential and the stationary radial equation of the spherically-symmetric boson star (interpreting the radial variable r as time). In [14] it was argued that Q-balls possess angular excitations as well and that the solutions should exist with an angular dependence in correspondence with the spherical harmonic functions $Y_\ell^k(\theta, \varphi)$. The gravitating counterparts of these excited solutions was addressed in [18],[24]. As pointed out in the previous sections, the cases $\ell = k = 0$ and $\ell = k = 1$ respectively correspond to spherically symmetric solutions

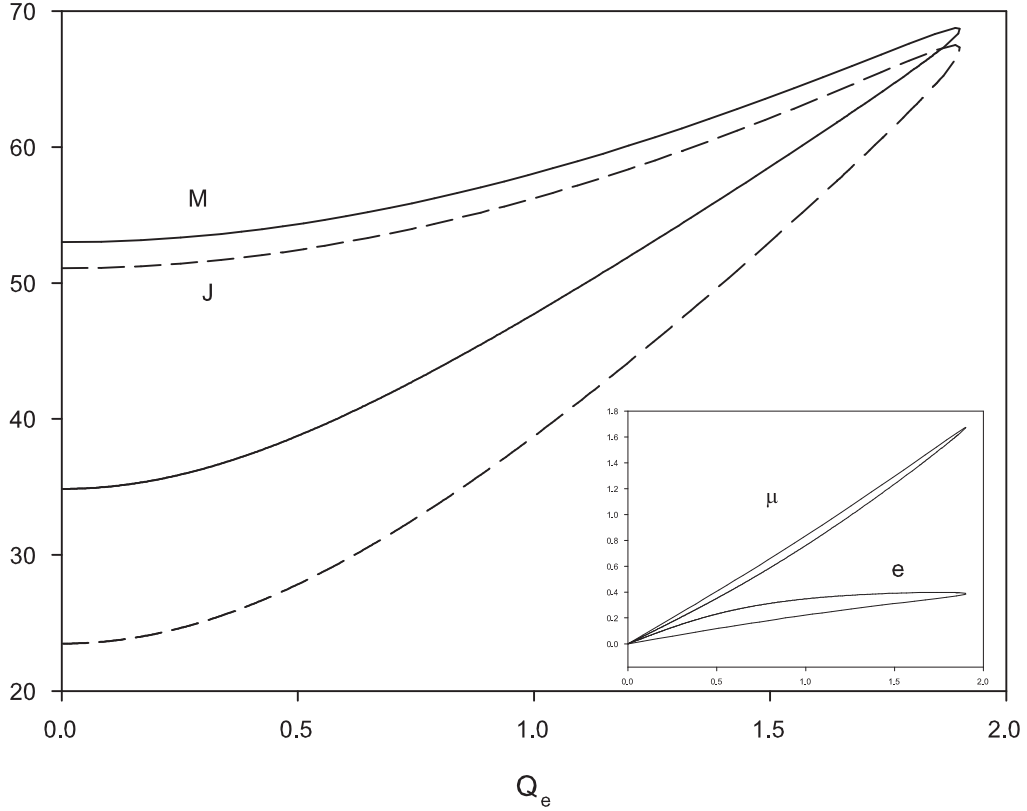


Figure 3: Mass M and angular momentum J of the $\ell = k = 1$ Q-ball as functions of the electric charge Q_e for $\alpha = 0.5$ and $\omega_s = 0.75$

and spinning solutions. The case $\ell = 2, k = 1$ corresponds to spinning solution with function ϕ antisymmetric under the reflexion $z \rightarrow -z$ which we discuss in the next section.

In [14] solutions corresponding to $\ell = 1, k = 0$ and $\ell = 2, k = 2$ have been constructed. Here we devote some attention to the charge and gravitating version of the $\ell = 1, k = 0$ excited solution. The angular dependance in θ goes roughly like $|Y_1^0| \propto \cos \theta$. Accordingly it corresponds to an antisymmetric configuration of the boson field and $|\phi|$ presents maxima at two opposite points along the oz axis. The electromagnetic current is even since it is propotionnal to the product $\phi\phi^*$. For $k = 0$, the equations for the fields w, A are trivially satisfied by $w = A = 0$; the solution is therefore non-spinning. The mass of this solution (for $\omega_s = 0.75$, $\alpha = 0.1$) is presented as a function of the charge on Fig.5. The contour plots of the solutions having $Q_e = 1$ are shown on Fig. 6. The maxima of the energy density are localized in two regions centered about the points $(x, y, z) = (0, 0, \pm 6.5)$. The functions f, V are even, revealing that the

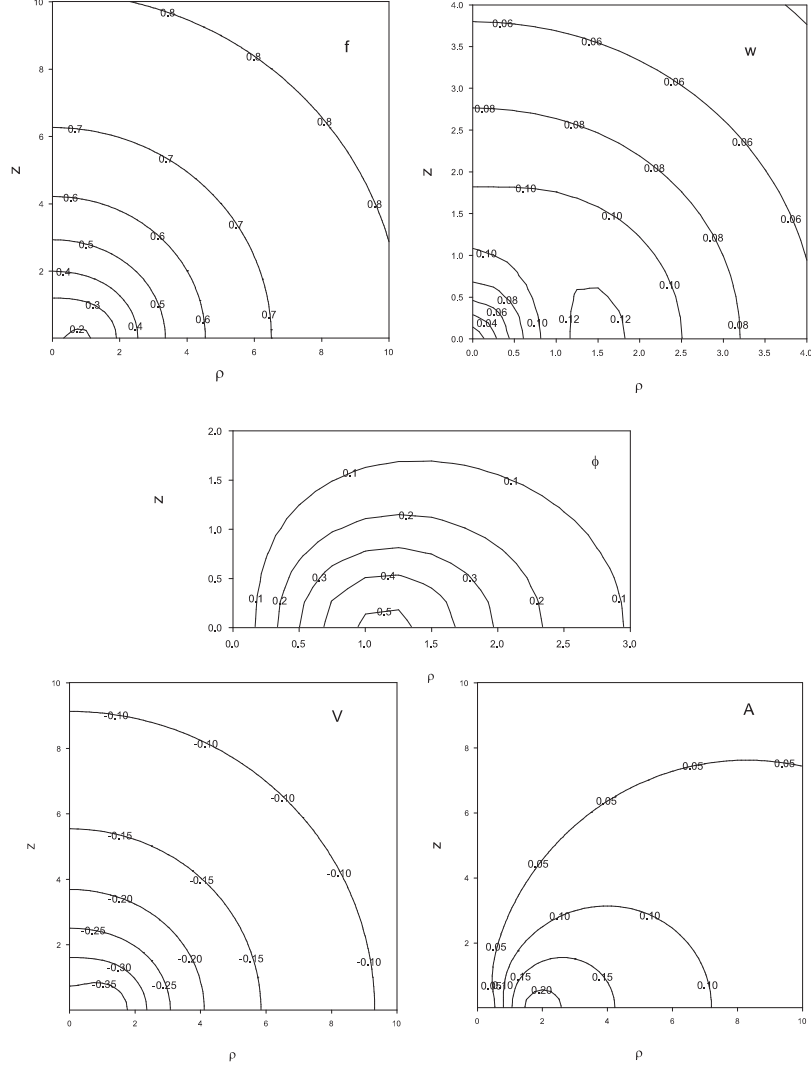


Figure 4: Contour plots of the functions f, w, ϕ, V and A for the $\ell = k = 1$ solution with $\alpha = 0.5$ $Q_e = 1$ and $\omega_s = 0.75$

two maxima of gravitating matter roughly coincide with two maxima of the density of electric charge.

4.4 Parity-odd Charged spinning boson stars

In the section above, we discussed the charged version of fundamental parity even spinning boson stars, it was noticed in [13] that spinning solutions with odd parity exist as well, their angular dependence is related to the Y_2^1 harmonic function, with $Y_2^1 \propto \sin \theta \cos \theta$. We succeeded

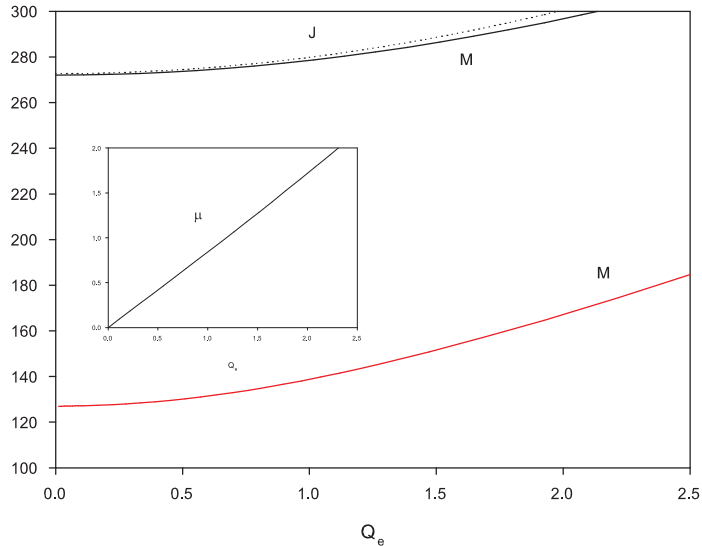


Figure 5: Mass of the $\ell = 0, k = 1$ solution as function of Q_e (red line). Mass, angular momentum (black lines) and magnetic moment (in the window) of the $\ell = 2, k = 1$ solution corresponding to $\alpha = 0.1$ and $\omega_s = 0.75$

in constructing the charged-gravitating version of this solution. The mass M and angular momentum J are reported on Fig.5 as functions of the electric charge by the black lines. The corresponding magnetic moment is reported in the window. The contour plot of such a solution is shown in Fig.7, we see in particular that the scalar field is maximal on a torus centered about the oz axis and located at $z \approx 2.5$. Correspondingly, the field is minimal on the mirror symmetric torus at $z \approx -2.5$. The electric and magnetic potentials are even under the reflexion $z \rightarrow -z$ and present extrema in the two tori. The topology of the vector potential A suggests the structure of a double magnetic dipole.

5 Conclusions

Numerical arguments suggesting the existence of several families of charged gravitating, axially symmetric Q-balls have been obtained. The singularities that occur naturally in the case of Einstein-Maxwell solutions are avoided by the scalar field, so that the solutions are regular on space-time. We have argued that families of charged solitons exist for which the angular dependance of the scalar field possesses the symmetries of the spherical harmonics $Y_\ell^k(\theta, \varphi)$. These stationary solutions can therefore be labelled by the integers ℓ, k .

Asymptotically, the space-time corresponding to the case $\ell = k = 0$ solution approaches the

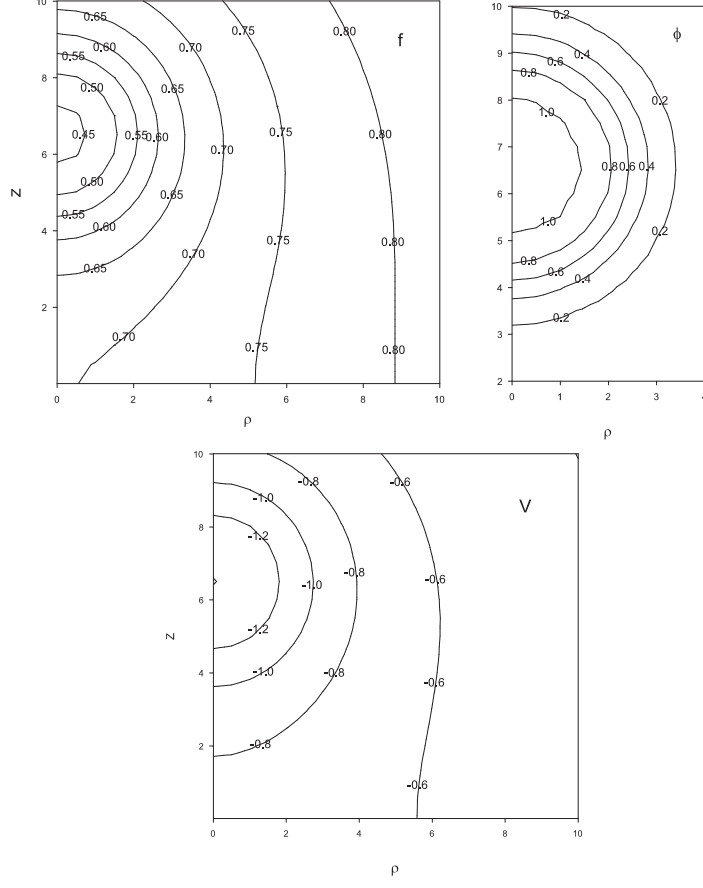


Figure 6: Contour plots of the functions f , ϕ and V for the $\ell = 1, k = 0$ solution for $\alpha = 0.5$, $\omega_s = 0.75$ and $Q_e = 1$

Reisner-Nordstrom black hole. In the case $\ell = k = 1$, the Q-balls create a Kerr-Neumann space-time asymptotically. The other angular excitations generates space-times which, asymptotically, present more involved angular dependances; it would be challenging to prove whether these solutions have (or not) black hole counterparts.

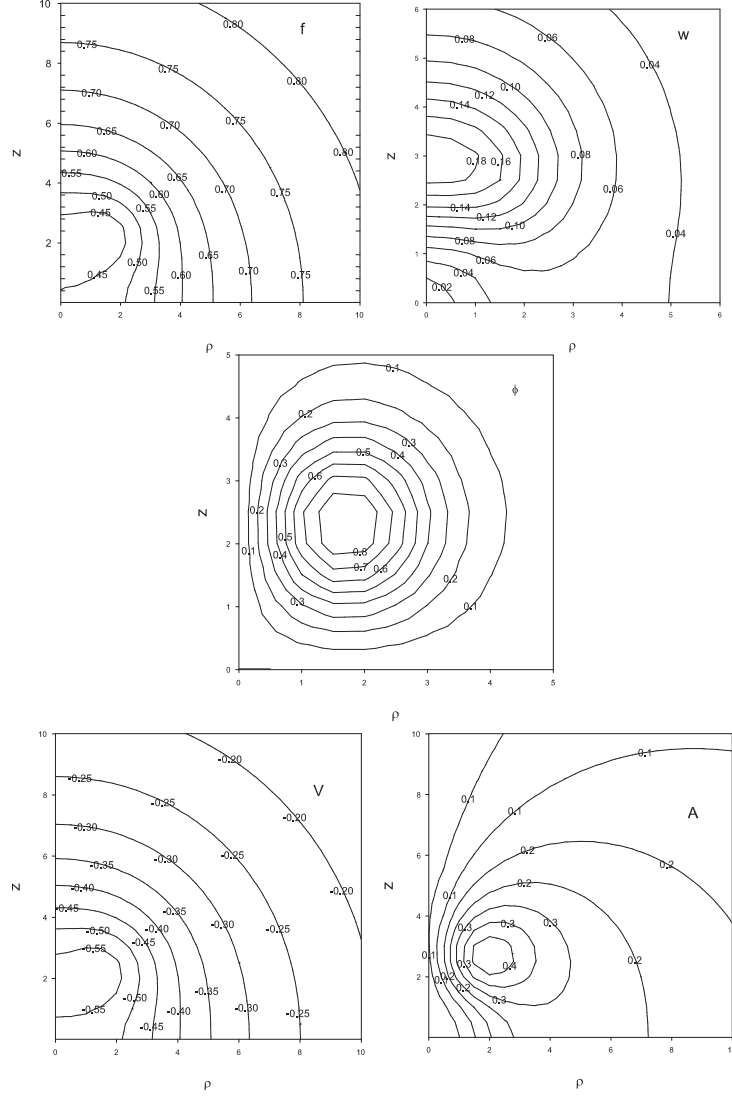


Figure 7: Contour plots of the functions f, w, ϕ, V and A for the $\ell = 2, k = 1$ solution for $\alpha = 0.1$, $\omega_s = 0.75$ and $Q_e = 1$

References

- [1] G. Jungman, M. Kamionkowski and G. Griest, Phys. Rept. **267** (1996) 195; G. Bertone, D. Hooper and J. Silk, *ibid*, **405** (2005) 279.
- [2] R. Friedberg, T. D. Lee and A. Sirlin, Phys. Rev. D **13** (1976) 2739.
- [3] T. D. Lee and Y. Pang, Phys. Rep. **221** (1992), 251.

- [4] S. R. Coleman, Nucl. Phys. B **262** (1985), 263.
- [5] E. Mielke and F. E. Schunck, Proc. 8th Marcel Grossmann Meeting, Jerusalem, Israel, 22-27 Jun 1997, World Scientific (1999), 1607.
- [6] R. Friedberg, T. D. Lee and Y. Pang, Phys. Rev. D **35** (1987), 3658.
- [7] P. Jetzer, Phys. Rept. **220** (1992), 163.
- [8] F. Cappella, R. Cerulli and A. Incicchitti, Eur. Phys. J. C **4** (2002) 14; Y. Takenaga *et al.* [Super-Kamookande collaboration], Phys. Lett. B **647** (2007) 18; S. Cecchini *et al.* [SLIM collaboration], Eur. Phys. J. C **57** (2008) 525.
- [9] G. R. Dvali, A. Kusenko and M. E. Shaposhnikov, Phys. Lett. B **417** (1998) 99.
- [10] A. Kusenko, Phys. Lett. B **404** (1997), 285; Phys. Lett. B **405** (1997), 108.
- [11] A. Kusenko and M. E. Shaposhnikov, Phys. Lett. B **418** (1998) 46.
- [12] *see e.g.* A. Kusenko, hep-ph/0009089.
- [13] M.S. Volkov and E. Wöhrner, Phys. Rev. D **66** (2002), 085003.
- [14] Y. Brihaye and B. Hartmann, Nonlinearity **21** (2008), 1937.
- [15] L. Campanelli and M. Ruggieri, Phys. Rev. D **77** (2008), 043504.
- [16] L. Campanelli and M. Ruggieri, "Spinning Supersymmetric Q-balls" arXiv:0904.4802.
- [17] B. Kleihaus, J. Kunz and M. List, Phys. Rev. D **72** (2005), 064002.
- [18] B. Kleihaus, J. Kunz, M. List and I. Schaffer, Phys. Rev. D **77** (2008), 064025.
- [19] E. Radu and M. Volkov, Phys. Rept. **468** (2008) 101.
- [20] B. Kleihaus, J. Kunz, C. Lämmerzahl and M. List, "Charged Boson Stars and Black Holes", arXiv:0902.4799.
- [21] S. Kasuya and F. Takahashi, Phys. Rev. D **72** (2005), 085015.
- [22] F. E. Schunck and D. F. Torres, Int. J. Mod. Phys. D **9** (2000), 601.
- [23] F. E. Schunck and E. Mielke, in *Relativity and Scientific Computing*, edited by F. W. Hehl, R. A. Putingam and H. Ruder (Springer, Berlin, 1996) 138.
- [24] Y. Brihaye and B. Hartmann, Phys. Rev. D **79** (2009), 064013.

- [25] W. Schönauer and R. Weiß, J. Comput. Appl. Math. **27** (1989) 279; M. Schauder, R. Weiß and W. Schönauer, “The CADSOL Program Package”, Universität Karlsruhe, Interner Bericht Nr. 46/92 (1992); W. Schönauer and E. Schnepf, ACM Trans. Math. Softw. **13** (1987) 333.

6 Appendix

In this appendix we give the field equations in the axially symmetric ansatz. The notation $F^{(i,j)} \equiv \frac{\partial^i}{\partial r^i} \frac{\partial^j}{\partial \theta^j} F$ is used.

The equations for the Maxwell fields read

$$\begin{aligned}
& \frac{V^{(0,2)}}{r^2} + V^{(2,0)} + \frac{2\omega A^{(0,1)} f^{(0,1)} \sin(\theta)}{f r^3} + \frac{2\omega A^{(1,0)} f^{(1,0)} \sin(\theta)}{f r} - \frac{L\omega^2 A^{(0,1)} \omega^{(0,1)} \sin^3(\theta)}{f^2 r^3} \\
& + \frac{L\omega^3 A^{(1,0)} \sin^3(\theta)}{f^2 r^2} - \frac{L\omega^2 A^{(1,0)} \omega^{(1,0)} \sin^3(\theta)}{f^2 r} - \frac{\omega A^{(0,1)} L^{(0,1)} \sin(\theta)}{L r^3} - \frac{\omega A^{(1,0)} L^{(1,0)} \sin(\theta)}{L r} \\
& - \frac{A^{(0,1)} \omega^{(0,1)} \sin(\theta)}{r^3} - \frac{2\omega A^{(0,1)} \cos(\theta)}{r^3} - \frac{\omega^{(1,0)} \sin(\theta)}{r^2} - \frac{A^{(1,0)} \omega^{(1,0)} \sin(\theta)}{r} \\
& + \frac{2A\omega f^{(0,1)} \cos(\theta)}{f r^3} - \frac{A L \omega^2 \omega^{(0,1)} \sin^2(\theta) \cos(\theta)}{f^2 r^3} - \frac{A \omega L^{(0,1)} \cos(\theta)}{L r^3} - \frac{A \omega^{(0,1)} \cos(\theta)}{r^3} \\
& - \frac{2e^2 L V \phi^2 g}{f} - \frac{2e L \phi^2 \omega_s g}{f} - \frac{f^{(0,1)} V^{(0,1)}}{f r^2} - \frac{f^{(1,0)} V^{(1,0)}}{f} + \frac{L \omega V^{(0,1)} \omega^{(0,1)} \sin^2(\theta)}{f^2 r^2} \\
& - \frac{L \omega^2 V^{(1,0)} \sin^2(\theta)}{f^2 r} + \frac{L \omega V^{(1,0)} \omega^{(1,0)} \sin^2(\theta)}{f^2} + \frac{L^{(0,1)} V^{(0,1)}}{2 L r^2} + \frac{L^{(1,0)} V^{(1,0)}}{2 L} \\
& + \frac{V^{(0,1)} \cot(\theta)}{r^2} + \frac{2 V^{(1,0)}}{r} - \frac{2 A \omega \cos(\theta) \cot(\theta)}{r^3} = 0,
\end{aligned} \tag{20}$$

$$\begin{aligned}
& \frac{A^{(0,2)}}{r^2} + A^{(2,0)} + \frac{A^{(0,1)} f^{(0,1)}}{f r^2} + \frac{A^{(1,0)} f^{(1,0)}}{f} - \frac{L \omega A^{(0,1)} \omega^{(0,1)} \sin^2(\theta)}{f^2 r^2} + \frac{L \omega^2 A^{(1,0)} \sin^2(\theta)}{f^2 r} \\
& - \frac{L \omega A^{(1,0)} \omega^{(1,0)} \sin^2(\theta)}{f^2} - \frac{A^{(0,1)} L^{(0,1)}}{2 L r^2} - \frac{A^{(1,0)} L^{(1,0)}}{2 L} + \frac{A^{(0,1)} \cot(\theta)}{r^2} \\
& - \frac{2 A e^2 L \phi^2 g}{f} + \frac{A f^{(0,1)} \cot(\theta)}{f r^2} - \frac{A L \omega \omega^{(0,1)} \sin(2\theta)}{2 f^2 r^2} - \frac{A L^{(0,1)} \cot(\theta)}{2 L r^2} + \frac{2 e L n \phi^2 \csc(\theta) g}{f} \\
& + \frac{L V^{(0,1)} \omega^{(0,1)} \sin(\theta)}{f^2 r} + \frac{L r V^{(1,0)} \omega^{(1,0)} \sin(\theta)}{f^2} - \frac{L \omega V^{(1,0)} \sin(\theta)}{f^2} - \frac{A \csc^2(\theta)}{r^2} = 0.
\end{aligned} \tag{21}$$

The scalar field equation reads

$$\begin{aligned}
& \frac{\phi^{(0,2)}}{r^2} + \phi^{(2,0)} - \frac{A^2 e^2 L \phi \omega^2 \sin^2(\theta) g}{f^2 r^2} + \frac{A^2 e^2 \phi g}{r^2} + \frac{2 A e^2 L V \phi \omega \sin(\theta) g}{f^2 r} \\
& + \frac{2 A e L n \phi \omega^2 \sin(\theta) g}{f^2 r^2} + \frac{2 A e L \phi \omega \omega_s \sin(\theta) g}{f^2 r} - \frac{2 A e n \phi \csc(\theta) g}{r^2} - \frac{e^2 L V^2 \phi g}{f^2} \\
& - \frac{2 e L n V \phi \omega g}{f^2 r} - \frac{2 e L V \phi \omega_s g}{f^2} - \frac{L n^2 \phi \omega^2 g}{f^2 r^2} - \frac{2 L n \phi \omega \omega_s g}{f^2 r} - \frac{L \phi \omega_s^2 g}{f^2} + \frac{L U'(\phi) g}{2 f} \\
& + \frac{n^2 \phi \csc^2(\theta) g}{r^2} - \frac{L^{(0,1)} \phi^{(0,1)}}{2 L r^2} - \frac{L^{(1,0)} \phi^{(1,0)}}{2 L} - \frac{\phi^{(0,1)} \cot(\theta)}{r^2} - \frac{2 \phi^{(1,0)}}{r} = 0.
\end{aligned} \tag{22}$$

The equations for the metric fields read

$$\begin{aligned}
& \frac{f^{(0,2)}}{r^2} + f^{(2,0)} - \frac{6Af^2\alpha A^{(0,1)} \cot(\theta)}{Lr^4} - \frac{3f^2\alpha (A^{(0,1)})^2}{Lr^4} - \frac{3f^2\alpha (A^{(1,0)})^2}{Lr^2} - \frac{\alpha\omega^2 (A^{(0,1)})^2 \sin^2(\theta)}{r^4} \\
& - \frac{2A\alpha\omega^2 A^{(0,1)} \sin(\theta) \cos(\theta)}{r^4} + \frac{2\alpha\omega A^{(0,1)} V^{(0,1)} \sin(\theta)}{r^3} - \frac{\alpha\omega^2 (A^{(1,0)})^2 \sin^2(\theta)}{r^2} + \frac{2\alpha\omega A^{(1,0)} V^{(1,0)} \sin(\theta)}{r} \\
& - \frac{4A^2 e^2 L \alpha \phi^2 \omega^2 \sin^2(\theta) g}{fr^2} + \frac{8Ae^2 LV \alpha \phi^2 \omega \sin(\theta) g}{fr} + \frac{8AeLn \alpha \phi^2 \omega^2 \sin(\theta) g}{fr^2} + \frac{8AeL \alpha \phi^2 \omega \omega_s \sin(\theta) g}{fr} \\
& + \frac{2A\alpha\omega V^{(0,1)} \cos(\theta)}{r^3} - \frac{4e^2 LV^2 \alpha \phi^2 g}{f} - \frac{8eLnV \alpha \phi^2 \omega g}{fr} - \frac{8eLV \alpha \phi^2 \omega_s g}{f} + \frac{f^{(0,1)} L^{(0,1)}}{2Lr^2} + \frac{f^{(1,0)} L^{(1,0)}}{2L} \\
& + \frac{f^{(0,1)} \cot(\theta)}{r^2} - \frac{(f^{(0,1)})^2}{fr^2} + \frac{2f^{(1,0)}}{r} - \frac{(f^{(1,0)})^2}{f} - \frac{4Ln^2 \alpha \phi^2 \omega^2 g}{fr^2} - \frac{8Ln \alpha \phi^2 \omega \omega_s g}{fr} - \frac{4L \alpha \phi^2 \omega_s^2 g}{f} \\
& - \frac{L (\omega^{(0,1)})^2 \sin^2(\theta)}{fr^2} + \frac{2L\omega \omega^{(1,0)} \sin^2(\theta)}{fr} - \frac{L (\omega^{(1,0)})^2 \sin^2(\theta)}{f} + 2L\alpha U(\phi)g - \frac{\alpha (V^{(0,1)})^2}{r^2} - \alpha (V^{(1,0)})^2 \\
& - \frac{3A^2 f^2 \alpha \cot^2(\theta)}{Lr^4} - \frac{A^2 \alpha \omega^2 \cos^2(\theta)}{r^4} - \frac{L\omega^2 \sin^2(\theta)}{fr^2} = 0,
\end{aligned} \tag{23}$$

$$\begin{aligned}
& \frac{L^{(0,2)}}{r^2} + L^{(2,0)} + \frac{2L\alpha\omega^2 (A^{(0,1)})^2 \sin^2(\theta)}{fr^4} + \frac{4AL\alpha\omega^2 A^{(0,1)} \sin(\theta) \cos(\theta)}{fr^4} - \frac{4L\alpha\omega A^{(0,1)} V^{(0,1)} \sin(\theta)}{fr^3} \\
& + \frac{2L\alpha\omega^2 (A^{(1,0)})^2 \sin^2(\theta)}{fr^2} - \frac{4L\alpha\omega A^{(1,0)} V^{(1,0)} \sin(\theta)}{fr} - \frac{4Af\alpha A^{(0,1)} \cot(\theta)}{r^4} - \frac{2f\alpha (A^{(0,1)})^2}{r^4} \\
& - \frac{2f\alpha (A^{(1,0)})^2}{r^2} - \frac{4A^2 e^2 L^2 \alpha \phi^2 \omega^2 \sin^2(\theta) g}{f^2 r^2} + \frac{4A^2 e^2 L \alpha \phi^2 g}{r^2} + \frac{8Ae^2 L^2 V \alpha \phi^2 \omega \sin(\theta) g}{f^2 r} \\
& + \frac{8AeL^2 n \alpha \phi^2 \omega^2 \sin(\theta) g}{f^2 r^2} + \frac{8AeL^2 \alpha \phi^2 \omega \omega_s \sin(\theta) g}{f^2 r} - \frac{8AeLn \alpha \phi^2 \csc(\theta) g}{r^2} - \frac{4AL\alpha\omega V^{(0,1)} \cos(\theta)}{fr^3} \\
& - \frac{4e^2 L^2 V^2 \alpha \phi^2 g}{f^2} - \frac{8eL^2 n V \alpha \phi^2 \omega g}{f^2 r} - \frac{8eL^2 V \alpha \phi^2 \omega_s g}{f^2} - \frac{4L^2 n^2 \alpha \phi^2 \omega^2 g}{f^2 r^2} - \frac{8L^2 n \alpha \phi^2 \omega \omega_s g}{f^2 r} \\
& - \frac{4L^2 \alpha \phi^2 \omega_s^2 g}{f^2} + \frac{4L^2 \alpha U(\phi) g}{f} + \frac{2L\alpha (V^{(0,1)})^2}{fr^2} + \frac{2L\alpha (V^{(1,0)})^2}{f} + \frac{4Ln^2 \alpha \phi^2 \csc^2(\theta) g}{r^2} \\
& + \frac{2L^{(0,1)} \cot(\theta)}{r^2} - \frac{(L^{(0,1)})^2}{2Lr^2} + \frac{3L^{(1,0)}}{r} - \frac{(L^{(1,0)})^2}{2L} + \frac{2A^2 L \alpha \omega^2 \cos^2(\theta)}{fr^4} - \frac{2A^2 f \alpha \cot^2(\theta)}{r^4} = 0,
\end{aligned} \tag{24}$$

$$\begin{aligned}
& \frac{g^{(0,2)}}{r^2} + g^{(2,0)} - \frac{2Afg\alpha A^{(0,1)} \cot(\theta)}{Lr^4} - \frac{fg\alpha (A^{(0,1)})^2}{Lr^4} - \frac{fg\alpha (A^{(1,0)})^2}{Lr^2} - \frac{3g\alpha\omega^2 (A^{(0,1)})^2 \sin^2(\theta)}{fr^4} \\
& - \frac{6Ag\alpha\omega^2 A^{(0,1)} \sin(\theta) \cos(\theta)}{fr^4} + \frac{6g\alpha\omega A^{(0,1)} V^{(0,1)} \sin(\theta)}{fr^3} - \frac{3g\alpha\omega^2 (A^{(1,0)})^2 \sin^2(\theta)}{fr^2} \\
& + \frac{6g\alpha\omega A^{(1,0)} V^{(1,0)} \sin(\theta)}{fr} + \frac{6Ag\alpha\omega V^{(0,1)} \cos(\theta)}{fr^3} - \frac{3gL (\omega^{(0,1)})^2 \sin^2(\theta)}{2f^2 r^2} + \frac{3gL\omega\omega^{(1,0)} \sin^2(\theta)}{f^2 r} \\
& - \frac{3gL (\omega^{(1,0)})^2 \sin^2(\theta)}{2f^2} + \frac{g (f^{(0,1)})^2}{2f^2 r^2} + \frac{g (f^{(1,0)})^2}{2f^2} - \frac{3g\alpha (V^{(0,1)})^2}{fr^2} \\
& - \frac{3g\alpha (V^{(1,0)})^2}{f} - \frac{(g^{(0,1)})^2}{gr^2} + \frac{g^{(1,0)}}{r} - \frac{(g^{(1,0)})^2}{g} - \frac{2gL^{(0,1)} \cot(\theta)}{Lr^2} - \frac{2gL^{(1,0)}}{Lr} \\
& - \frac{g (L^{(0,1)})^2}{2L^2 r^2} - \frac{g (L^{(1,0)})^2}{2L^2} + \frac{2g\alpha (\phi^{(0,1)})^2}{r^2} + 2g\alpha (\phi^{(1,0)})^2 + \frac{2A^2 e^2 g^2 L\alpha\phi^2\omega^2 \sin^2(\theta)}{f^2 r^2} \\
& - \frac{6A^2 e^2 g^2 \alpha\phi^2}{r^2} - \frac{A^2 fg\alpha \cot^2(\theta)}{Lr^4} - \frac{3A^2 g\alpha\omega^2 \cos^2(\theta)}{fr^4} - \frac{4Ae^2 g^2 LV\alpha\phi^2\omega \sin(\theta)}{f^2 r} - \frac{4Aeg^2 Ln\alpha\phi^2\omega^2 \sin(\theta)}{f^2 r^2} \\
& - \frac{4Aeg^2 L\alpha\phi^2\omega\omega_s \sin(\theta)}{f^2 r} + \frac{12Aeg^2 n\alpha\phi^2 \csc(\theta)}{r^2} + \frac{2e^2 g^2 LV^2\alpha\phi^2}{f^2} + \frac{4eg^2 LnV\alpha\phi^2\omega}{f^2 r} + \frac{4eg^2 LV\alpha\phi^2\omega_s}{f^2} \\
& + \frac{2g^2 Ln^2\alpha\phi^2\omega^2}{f^2 r^2} + \frac{4g^2 Ln\alpha\phi^2\omega\omega_s}{f^2 r} + \frac{2g^2 L\alpha\phi^2\omega_s^2}{f^2} - \frac{3gL\omega^2 \sin^2(\theta)}{2f^2 r^2} - \frac{2g^2 L\alpha U(\phi)}{f} - \frac{6g^2 n^2\alpha\phi^2 \csc^2(\theta)}{r^2} = 0,
\end{aligned} \tag{25}$$

$$\begin{aligned}
& \frac{\omega^{(0,2)}}{r^2} + \omega^{(2,0)} - \frac{8Afa\omega A^{(0,1)} \cot(\theta)}{Lr^4} - \frac{4fa\omega (A^{(0,1)})^2}{Lr^4} + \frac{4faA^{(0,1)} V^{(0,1)} \csc(\theta)}{Lr^3} - \frac{4fa\omega (A^{(1,0)})^2}{Lr^2} \\
& + \frac{4faA^{(1,0)} V^{(1,0)} \csc(\theta)}{Lr} - \frac{4A^2 e^2 \alpha\phi^2\omega g}{r^2} + \frac{4Ae^2 V\alpha\phi^2 \csc(\theta)g}{r} + \frac{8Aen\alpha\phi^2\omega \csc(\theta)gg(r, \theta)}{r^2} \\
& + \frac{4Aea\phi^2\omega_s \csc(\theta)g}{r} + \frac{4AfaV^{(0,1)} \cot(\theta) \csc(\theta)}{Lr^3} - \frac{4enV\alpha\phi^2 \csc^2(\theta)g}{r} - \frac{2f^{(0,1)}\omega^{(0,1)}}{fr^2} \\
& + \frac{2\omega f^{(1,0)}}{fr} - \frac{2f^{(1,0)}\omega^{(1,0)}}{f} - \frac{4n^2\alpha\phi^2\omega \csc^2(\theta)g}{r^2} - \frac{4n\alpha\phi^2\omega_s \csc^2(\theta)g}{r} + \frac{3L^{(0,1)}\omega^{(0,1)}}{2Lr^2} \\
& - \frac{3\omega L^{(1,0)}}{2Lr} + \frac{3L^{(1,0)}\omega^{(1,0)}}{2L} + \frac{3\omega^{(0,1)} \cot(\theta)}{r^2} + \frac{2\omega^{(1,0)}}{r} - \frac{4A^2 fa\omega \cot^2(\theta)}{Lr^4} - \frac{2\omega}{r^2} = 0,
\end{aligned} \tag{26}$$